

Spanning trees and line graph eigenvalues

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Abstract

In this note, we prove if G is a graph with an odd number of vertices and an odd number of spanning trees, then the adjacency matrix of the line graph of G is nonsingular. We also show that if a connected graph G has $2^t s$ spanning trees with s odd, then the nullity of the adjacency matrix of the line graph of G is at most $t + 1$.

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1 Introduction

By the eigenvalues of a graph we mean the eigenvalues of its adjacency matrix. A graph is said to be singular if its adjacency matrix is singular. Similarly, the nullity of a graph means the nullity of its adjacency matrix.

In 2001, Gutman and Sciriha [5] proved that the nullity of the line graph $\mathcal{L}(T)$ of any tree T is at most one and if $\mathcal{L}(T)$ is singular, then T has an even number of vertices. Recently, Bapat [1] found an interesting generalization of this result by proving that if a graph G has an odd number of spanning trees, then $\mathcal{L}(G)$ has nullity at most one. He also showed that if $\mathcal{L}(G)$ is singular and G is bipartite with an odd number of spanning trees, then the number of vertices of G is even. We extend these results to the following.

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Theorem 1. *Let G be a connected graph with $2^t s$ spanning trees with s odd. Then the multiplicity of any even integer $\lambda \neq -2$ as an eigenvalue of $\mathcal{L}(G)$ is at most $t + 1$.*

Theorem 2. *Let G be a graph with odd order and an odd number of spanning trees. If an even integer $\lambda \neq -2$ is an eigenvalue of the line graph of G , then $\lambda \equiv 2 \pmod{4}$, λ is a simple eigenvalue, and $\mathcal{L}(G)$ has at most one such eigenvalue.*

Corollary 3. *If a graph G has odd order and odd number of spanning trees, then the line graph of G is nonsingular.*

2 Preliminaries

The graphs we consider are simple, that is, without loops or multiple edges. Let G be graph of order n . We denote by X the $0, 1$ vertex-edge incidence matrix of G . If we orient each edge of G , then D will denote the $0, \pm 1$ vertex-edge incidence matrix of the resulting graph. The *Laplacian matrix* of G is $L = L(G) = DD^\top$ and the *signless Laplacian* matrix of G is $Q = Q(G) = XX^\top$. Note that the Laplacian does not depend on the orientation. The matrices L and Q are positive semidefinite. The nullity of L and Q are respectively equal to the number of components and to the number of bipartite components of G . In fact, L and Q are similar if and only if G is bipartite (cf. [3]). Let

$$p_Q(x) = x^n + q_1 x^{n-1} + \cdots + q_n, \quad p_L(x) = x^n + \ell_1 x^{n-1} + \cdots + \ell_{n-1} x$$

be the characteristic polynomials of Q and L , respectively. A subgraph of G whose components are trees or unicyclic graphs with odd cycles is called a TU-subgraph of G . Suppose that a TU-subgraph H of G contain c unicyclic graphs and trees T_1, T_2, \dots, T_s . Then the weight $W(H)$ of H is defined by

$$W(H) = 4^c \prod_{i=1}^s (1 + e(T_i)),$$

where $e(T_i)$ denotes the number of edges of T_i . The weight of an acyclic subgraph, that is, a union of trees, is defined similarly with $c = 0$. We shall express the coefficients of $p_Q(x)$ and $p_L(x)$ in terms of the weights of TU-subgraphs and acyclic subgraphs of G .

Let $\tau(G)$ denote the number of spanning trees of G . By the Matrix-Tree Theorem, for all $1 \leq i, j \leq n$, $(-1)^{i+j} \det(L_{ij}) = \tau(G)$ where L_{ij} is the submatrix of L obtained by eliminating the i th row and j th column. It follows that $\ell_{n-1} = (-1)^{n-1} n \tau(G)$. The first part of the following theorem which is the generalization of the Matrix-Tree Theorem was appeared in [6] (see also [3]). The second part was proved in [4] (see also [2]).

Theorem 4. *The coefficients of $p_L(x)$ and $p_Q(x)$ are determined as follows.*

- (i) $\ell_j = (-1)^j \sum_{F_j} W(F_j)$, for $j = 1, \dots, n-1$, where the summation runs over all acyclic subgraphs F_j of G with j edges.
- (ii) $p_j = (-1)^j \sum_{H_j} W(H_j)$, for $j = 1, \dots, n$, where the summation runs over all TU -subgraphs H_j of G with j edges.

3 Proofs

If \mathcal{A} is the adjacency matrix of $\mathcal{L}(G)$, then $\mathcal{A} + 2I = X^\top X$. This implies that the matrices $\mathcal{A} + 2I$ and Q have the same nonzero eigenvalues with the same multiplicities. We prove the following equivalent statements of Theorem 1 and Theorem 2.

Theorem 5. *Let G be a connected graph having $2^t s$ spanning trees with s odd. Then the multiplicity of any even integer λ as an eigenvalue of $Q(G)$ is at most $t + 1$.*

Proof. It is well known that for a given integral matrix A of rank r , there exist unimodular matrices (that is, integral matrices with determinant ± 1) U and V such that

$$UAV = \text{diag}(s_1, \dots, s_r, 0, \dots, 0)$$

where s_1, \dots, s_r are positive integers with $s_1 s_2 \cdots s_i = d_i$ where d_i is the greatest common divisor of all minors of A of order i , $1 \leq i \leq r$. (The matrix $\text{diag}(s_1, \dots, s_r, 0, \dots, 0)$ is called the Smith form of A .)

Let $S = \text{diag}(s_1, \dots, s_{n-1}, 0)$ be the Smith form of L . Note that $\text{rank}_{\mathbb{Z}_2}(L) = \text{rank}_{\mathbb{Z}_2}(S)$. By the Matrix-Tree Theorem, $\tau(G) = d_{n-1} = s_1 s_2 \cdots s_{n-1}$. It follows that at most t of the s_i are even. Therefore, $\text{rank}_{\mathbb{Z}_2}(L) \geq n - t - 1$ and so $\text{rank}_{\mathbb{Z}_2}(Q) \geq n - t - 1$. It is known that any symmetric matrix of rank r (over any field) has a principal $r \times r$ submatrix of full rank. Hence Q has a full-rank principle submatrix B of order $n - t - 1$. By interlacing, if an even integer λ is an eigenvalue of Q with multiplicity at least $t + 2$, then any principle submatrix of Q of order $n - t - 1$ has λ as an eigenvalue. So λ is an eigenvalue of B . This implies that $\det(B)/\lambda$ is a rational algebraic integer and thus an integer. Hence $\det(B)$ is even, a contradiction. This completes the proof. \square

Theorem 6. *Let G be a graph with odd order and an odd number of spanning trees. If an even integer λ is an eigenvalue of $Q(G)$, then $\lambda \equiv 0 \pmod{4}$, λ is a simple eigenvalue, and $Q(G)$ has at most one such eigenvalue.*

Proof. Let G be of order n . From Theorem 4 it follows that for $j = 1, \dots, n-1$, $p_j = \ell_j + 4s_j$ for some integer s_j and $p_n = 4s_n$. This implies that $p_Q(x) = p_L(x) + 4f(x)$ where $f(x)$ is an integer

polynomial. Note that $\ell_{n-1} = (-1)^{n-1}n\tau(G)$ is an odd integer. It follows that if k is an integer congruent to 2 modulo 4, then so is $p_Q(k)$. This proves the first part of the theorem.

Let k , an even integer, be an eigenvalue of Q . Suppose that $\mathbf{x} = (x_1, \dots, x_n)^\top$ is an eigenvector corresponding to k . We may take \mathbf{x} to be an integral vector with its components relatively prime. Assume that \mathbf{x} , reduced modulo 2, has a zero coordinate, x_1 say. If $\mathbf{y} = (x_2, \dots, x_n)^\top$, then \mathbf{y} is a null vector for $Q_{11} - kI$ modulo 2 which is a contradiction because $\det(Q_{11} - kI)$ is congruent to $\det(L_{11}) = \tau(G)$ modulo 2. Thus, each component of \mathbf{x} is an odd integer. From the above argument we conclude that if all eigenvectors corresponding to an eigenvalue λ has no zero components, then λ is a simple eigenvalue. It turns out that k is a simple eigenvalue of Q . If $k' \neq k$ is also an even eigenvalue of Q , then k' possesses an eigenvector \mathbf{x}' such that each component of \mathbf{x} is an odd integer. Since n is odd, \mathbf{x} and \mathbf{x}' cannot be orthogonal, a contradiction. \square

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